L^1–SPECTRUM OF BANACH SPACE VALUED ORNSTEIN–UHLENBECK OPERATORS

ROSTYSLAV V. KOZHAN

Abstract. We characterize the $L^1(E,\mu_\infty)$–spectrum of the Ornstein–Uhlenbeck operator $Lf(x) = \frac{1}{2}\operatorname{Tr} QD^2 f(x) + \langle Ax, Df(x)\rangle$, where μ_∞ is the invariant measure for the Ornstein–Uhlenbeck semigroup generated by L. The main result covers the general case of an infinite-dimensional Banach space E under the assumption that the point spectrum of A^* is nonempty and extends several recent related results.

1. Introduction and results

In this paper we investigate spectral properties of the generator of the transition semigroup associated with the stochastic linear Cauchy problem

$$\begin{cases} dU_t &= AU_t dt + B dW_t^H, \\ U_0 &= x, \end{cases} \tag{1.1}$$

where A is the generator of a C_0-semigroup $(S(t))_{t\ge 0}$ on a real Banach space E, B is a nonzero bounded operator from a real Hilbert space H into E, $(W_t^H)_{t\ge0}$ is an H-cylindrical Wiener process, and $x \in E$. The problem is a natural infinite-dimensional generalization of the Langevin equation and arises in many applications, for example in optimal control theory and interest rate models, see [5, 6].

It is well known [1, 6] that the problem (1.1) admits a unique weak solution $\mathbf{U} = (U_t(x))_{t\ge0}$ if and only if for all $t \in (0,\infty)$ there exists a centered Gaussian Radon measure μ_t on E with covariance operator $Q_t \in \mathcal{L}(E^*, E)$ given by

$$\langle Q_t x^*, y^*\rangle = \int_0^t \langle S(s)BB^*S^*(s)x^*, y^*\rangle\, ds, \qquad x^*, y^* \in E^*, \tag{1.2}$$

and in this case the solution can be represented in the form

$$U_t(x) = S(t)x + \int_0^t S(t-s)B\, dW_t^H. \tag{1.3}$$

The process $\mathbf{U}$ in (1.3) is Gaussian and Markov and its transition semigroup $\mathbf{P} = (P(t))_{t\ge0}$ (the Ornstein–Uhlenbeck semigroup) is defined on bounded Borel functions on E by

$$(P(t)f)(x) := \mathbb{E}\left(f(U_t(x))\right) = \int_E f(S(t)x+y)\, d\mu_t(y).$$

2000 *Mathematics Subject Classification.* 47A10, 47D06, 60H15.

Assume that the limit $Q_\infty := \lim_{t\to\infty} Q_t$ exists in the weak operator topology of $\mathcal{L}(E^*, E)$ and that there exists a centered Gaussian Radon measure μ_∞ with covariance operator Q_∞. Under this assumption the measure μ_∞ is invariant for $\mathbf{P}$ (see [1, 6]), i.e.

$$\int_E P(t)f(x)\, d\mu_\infty(x) = \int_E f(x)\, d\mu_\infty(x), \qquad t \geq 0.$$

Throughout the paper we assume μ_∞ to be nondegenerate.

The above equality easily implies (see [13, Thm XIII.1]) that the semigroup $\mathbf{P}$ has a unique extension to a strongly continuous contraction semigroup on $L^p(E, \mu_\infty)$, $p \in [1, \infty)$, which we also denote by $\mathbf{P}$. Properties of $\mathbf{P}$ in $L^p(E, \mu_\infty)$ for $p \in (1, \infty)$ have been extensively investigated in the literature (see [3, 4, 5, 9, 10] and references therein). Properties of $\mathbf{P}$ in $L^1(E, \mu_\infty)$ turn out to be completely different from those in the spaces $L^p(E, \mu_\infty)$, $p \in (1, \infty)$. In particular for $p = 1$ the semigroup $\mathbf{P}$ loses its regularity properties, which it possesses in the case $p \in (1, \infty)$ (see [4, 5, 8]), and the spectrum of its generator, which is p-independent for $p \in (1, \infty)$ (see [9, 10]), changes drastically.

The key issue investigated in the present paper is the structure of the spectrum of the generator L, called the Ornstein–Uhlenbeck operator, of this semigroup in $L^1(E, \mu_\infty)$. Denote by $\mathcal{C}(L)$ the set of continuous on E functions of the form $f(x) = \phi(\langle x, x_1^*\rangle, \dots, \langle x, x_n^*\rangle)$, where $x_j^* \in \mathcal{D}(A^*)$ for all $j = 1, \dots, n$ and $\phi \in C^2(\mathbb{R}^n)$ with compact support. It follows from [8] that $\mathcal{C}(L)$ is a core for L, and for $f \in \mathcal{C}(L)$,

$$Lf(x) = \frac{1}{2} \operatorname{Tr} D_H^2 f(x) + \langle Ax, Df(x)\rangle, \tag{1.4}$$

where $Df : E \to E^*$ is the Fréchet derivative of f, and $D_H f : E \to H^*$ is the Fréchet derivative of f in the direction of H defined by

$$D_H f(x) = \sum_{j=1}^n \frac{\partial \phi}{\partial x_j}(\langle x, x_1^*\rangle, \dots, \langle x, x_n^*\rangle) B^* x_j^*.$$

Denote $Q = BB^* \in \mathcal{L}(E^*, E)$. When E is a Hilbert space itself, the first term on the right-hand side of (1.4) becomes just $\frac{1}{2} \operatorname{Tr} QD^2 f(x)$.

By the result in [9, Thm 5.1], if $E = \mathbb{R}^n$ the $L^1(\mathbb{R}^n, \mu_\infty)$–spectrum of L is equal to $\overline{\mathbb{C}}_- = \{\lambda : \operatorname{Re}\lambda \leq 0\}$ with each $\lambda \in \mathbb{C}_- = \{\lambda : \operatorname{Re}\lambda < 0\}$ being an eigenvalue. This result was extended to infinite-dimensional Banach spaces E in [12] under the assumption of eventual compactness of the semigroup $(S(t))_{t\geq 0}$, and in [2] under the assumption that the part of A^* in the reproducing kernel Hilbert space of μ_∞ has an eigenvalue $\gamma \in \mathbb{C}_-$. Theorem 1.1 of the present paper generalizes both of these results while requiring less effort to prove it. Also it can serve as an alternative simple coordinate-free proof of the corresponding finite-dimensional result of [9].

Theorem 1.1. *If the point spectrum of A^* is not empty $\sigma_p(A^*) \neq \varnothing$, then the spectrum of the Ornstein–Uhlenbeck operator L coincides with $\overline{\mathbb{C}}_-$, and each $\lambda \in \mathbb{C}_-$ is its eigenvalue.*

We remark that due to [11, Pr 2.5] we have $\sigma_p(A^*) \cap \{\lambda : \operatorname{Re}\lambda \geq 0\} = \varnothing$, and thus the condition $\sigma_p(A^*) \neq \varnothing$ of Theorem 1.1 is equivalent to the existence of an eigenvalue of A^*

with negative real part. An extension of Theorem 1.1 for the case $\sigma_p(A^*) = \varnothing$ seems to be an open question so far.

Corollary 1.2. *Under the assumptions of Theorem 1.1, the Ornstein-Uhlenbeck semigroup $(P(t))_{t\geq 0}$ on $L^1(E, \mu_\infty)$ is norm discontinuous everywhere.*

Proof. Indeed, it easily follows from [7, Thm 4.18] that the spectrum of the generator of an eventually norm continuous semigroup cannot be equal to $\overline{\mathbb{C}}_-$. $\square$

Acknowledgement. The author would like to thank J.M.A.M. van Neerven and Rostyslav O. Hryniv for helpful remarks.

2. Proof of Theorem 1.1

That each $\lambda \in \mathbb{C}_-$ is an eigenvalue of L we establish in Lemmas 2.1 and 2.3. This implies $\mathbb{C}_- \subset \sigma_p(L) \subset \sigma(L)$. The fact that $\mathbf{P}$ is contractive on $L^1(E, \mu_\infty)$ implies $\sigma(L) \subset \overline{\mathbb{C}}_-$. Since the spectrum is closed, this finishes the proof.

By the argument in Section 1, we may assume that A^* has an eigenvalue $\gamma \in \mathbb{C}_-$. Denote the corresponding eigenvector as $x_0^* \in E^*_{\mathbb{C}}$, where $E^*_{\mathbb{C}}$ is the complexification of E^*.

Lemma 2.1. *If $\gamma \in \mathbb{R} \cap \mathbb{C}_-$, then each $\lambda \in \mathbb{C}_-$ is an eigenvalue of L.*

Proof. Since $\gamma \in \mathbb{R}$, the corresponding eigenvector x_0^* of A^* may be chosen in E^*. We will show that for each $\lambda \in \mathbb{C}_-$ we can find an eigenfunction of L of the form $f_\lambda(x) = \phi_\lambda(\langle x, x_0^*\rangle) : E \to \mathbb{C}$ with some function ϕ_λ on $\mathbb{R}$.

Consider the one-dimensional Ornstein–Uhlenbeck operator defined by

$$L_1\phi(t) = \frac{1}{2}q\phi''(t) + \gamma t\phi'(t)$$

for $\phi \in \mathcal{C}(L_1) = \{\phi \in C^2(\mathbb{R}) \text{ with compact support}\}$, with $q = \langle Qx_0^*, x_0^*\rangle \geq 0$. In fact $\langle Qx_0^*, x_0^*\rangle \neq 0$: x_0^* is an eigenvector of S^*, so (1.2) and $\langle Qx_0^*, x_0^*\rangle = 0$ imply $\langle Q_\infty x_0^*, x_0^*\rangle = 0$ which contradicts the assumption that μ_∞ is nondegenerate. Hence $q > 0$.

Here $\mathcal{C}(L_1)$ is viewed as a subspace of $L^1(\mathbb{R}, \nu_\infty)$, where ν_∞ is the invariant measure for L_1. Now observe that $\phi(t) \in L^1(\mathbb{R}, \nu_\infty)$ is equivalent to $\phi(\langle x, x_0^*\rangle) \in L^1(E, \mu_\infty)$. Indeed, ν_∞ is a centered one-dimensional Gaussian measure with variance $\int_0^\infty e^{\gamma s}qe^{\gamma s}ds = -\frac{1}{2\gamma}\langle Qx_0^*, x_0^*\rangle$. By definition, measure μ_∞ on the cylindrical function $\phi(\langle x, x_0^*\rangle)$ is centered one-dimensional Gaussian with variance

$$\langle Q_\infty x_0^*, x_0^*\rangle = \int_0^\infty \langle S(s)QS^*(s)x_0^*, x_0^*\rangle ds = \int_0^\infty \langle Qe^{\gamma s}x_0^*, e^{\gamma s}x_0^*\rangle ds = -\frac{1}{2\gamma}\langle Qx_0^*, x_0^*\rangle.$$

Hence $\phi(t) \in L^1(\mathbb{R}, \nu_\infty)$ if and only if $\phi(\langle x, x_0^*\rangle) \in L^1(E, \mu_\infty)$, and $L^1(\mathbb{R}, \nu_\infty)$-convergence is equivalent to $L^1(E, \mu_\infty)$-convergence for the corresponding $\langle x, x_0^*\rangle$-cylindrical functions.

Now, for $\phi \in \mathcal{C}(L_1)$, we have $f(x) := \phi(\langle x, x_0^*\rangle) \in \mathcal{C}(L)$ and

$$\langle Ax, Df(x)\rangle = \langle Ax, \phi'(\langle x, x_0^*\rangle)\, x_0^*\rangle = \phi'(\langle x, x_0^*\rangle)\, \langle x, A^*x_0^*\rangle = \gamma\, \phi'(\langle x, x_0^*\rangle)\, \langle x, x_0^*\rangle,$$

$$\left(D_H^2 f(x)\right)(y) = \phi''(\langle x, x_0^*\rangle)\, \langle y, B^*x_0^*\rangle\, B^*x_0^*, \quad y \in H.$$

The only nonzero eigenvalue of the operator on the right-hand side of the last equality is $\phi''(\langle x, x_0^*\rangle)\langle Qx_0^*, x_0^*\rangle$, so

$$Lf(x) = \frac{1}{2}\langle Qx_0^*, x_0^*\rangle\, \phi''(\langle x, x_0^*\rangle) + \gamma\, \langle x, x_0^*\rangle\, \phi'(\langle x, x_0^*\rangle) = (L_1\phi)\,(\langle x, x_0^*\rangle). \tag{2.1}$$

For each $\lambda \in \mathbb{C}_-$ let $\phi_\lambda(t) \in L^1(\mathbb{R}, \nu_\infty)$ be an eigenfunction of L_1 corresponding to the eigenvalue λ (we use the one-dimensional case of [9, Thm 5.1]). Now we find $\phi_n(t) \in \mathcal{C}(L_1)$ with $\phi_n \to \phi_\lambda$ in $L^1(\mathbb{R}, \nu_\infty)$. Then $\phi_n(\langle x, x_0^*\rangle) \in \mathcal{C}(L)$ and $\phi_n(\langle x, x_0^*\rangle) \to \phi_\lambda(\langle x, x_0^*\rangle)$ in $L^1(E, \mu_\infty)$, and applying (2.1) to $\phi_n(\langle x, x_0^*\rangle)$'s we obtain $f_\lambda(x) := \phi_\lambda(\langle x, x_0^*\rangle) \in \mathcal{D}(L)$ with $Lf_\lambda(x) = \lambda f_\lambda(x)$.

□

We state an easy auxiliary lemma, the proof of which is left to the reader.

Lemma 2.2. *Let H be a Hilbert space, $x_1, x_2, y_1, y_2 \in H$. The trace of the operator $Ax = \langle x, y_1\rangle x_1 + \langle x, y_2\rangle x_2$ is equal to* $\operatorname{Tr} A = \langle x_1, y_1\rangle + \langle x_2, y_2\rangle$.

Now we prove

Lemma 2.3. *Let $\gamma \in \mathbb{C}_- \setminus \mathbb{R}$. Then each $\lambda \in \mathbb{C}_-$ is an eigenvalue of L.*

Proof. Let $\gamma = a + bi$, $a < 0$, $b \neq 0$. Take $h_1^* := \operatorname{Re} x_0^* \in E^*$, $h_2^* := \operatorname{Im} x_0^* \in E^*$. We have $A^* h_1^* = ah_1^* - bh_2^*$, $A^* h_2 = bh_1^* + ah_2^*$, and also

$$S^*(s)h_1^* = e^{as}(h_1^* \cos bs - h_2^* \sin bs), \tag{2.2}$$

$$S^*(s)h_2^* = e^{as}(h_1^* \sin bs + h_2^* \cos bs). \tag{2.3}$$

We follow the same approach as in Lemma 2.1. Consider the two-dimensional Ornstein–Uhlenbeck operator

$$L_2\phi(t) := \frac{1}{2}\operatorname{Tr}(RD^2\phi(t)) + \langle Ct, D\phi(t)\rangle, \qquad t \in \mathbb{R}^2$$

for $\phi \in \mathcal{C}(L_2) = \{\phi \in C^2(\mathbb{R}^2)$ with compact support$\}$, where

$$R := (r_{ij})_{i,j=1}^2 = (\langle Qh_i^*, h_j^*\rangle)_{i,j=1}^2, \quad C := (c_{ij})_{i,j=1}^2 = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

Now we show that the covariance operator R_∞ of the invariant measure ν_∞ corresponding to L_2 is the same as of the image measure of μ_∞ under the map $(\langle x, h_1^*\rangle, \langle x, h_2^*\rangle)$. This can be verified directly: using (2.2) and (2.3), it is easy to show that the matrix $e^{Cs}Re^{C^*s}$ equals to the matrix $(\langle S(s)QS^*(s)h_i^*, h_j^*\rangle)_{i,j=1}^2$, and thus

$$\begin{aligned} R_\infty = \int_0^\infty e^{Cs}Re^{C^*s}\, ds &= \left(\int_0^\infty \langle S(s)QS^*(s)h_i^*, h_j^*\rangle\, ds\right)_{i,j=1}^2 \\ &= \left(\langle Q_\infty h_i^*, h_j^*\rangle\right)_{i,j=1}^2 . \end{aligned} \tag{2.4}$$

This implies that $\phi(t) \in L^1(\mathbb{R}^2, \nu_\infty)$ if and only if $\phi(\langle x, h_1^*\rangle, \langle x, h_2^*\rangle) \in L^1(E, \mu_\infty)$, and $L^1(\mathbb{R}^2, \nu_\infty)$-convergence is equivalent to $L^1(E, \mu_\infty)$-convergence for the corresponding $(\langle x, h_1^*\rangle, \langle x, h_2^*\rangle)$-cylindrical functions.

Now, for $\phi \in \mathcal{C}(L_2)$, we have $f(x) := \phi(\langle x, h_1^*\rangle, \langle x, h_2^*\rangle) \in \mathcal{C}(L)$ and

$$\langle Ax, Df(x)\rangle = \langle Ax, \frac{\partial \phi}{\partial t_1}(\langle x, h_1^*\rangle, \langle x, h_2^*\rangle)\ h_1^*\rangle + \langle Ax, \frac{\partial \phi}{\partial t_2}(\langle x, h_1^*\rangle, \langle x, h_2^*\rangle)\, h_2^*\rangle$$
$$= \frac{\partial \phi}{\partial t_1}(\langle x, h_1^*\rangle, \langle x, h_2^*\rangle)\,(a\langle x, h_1^*\rangle - b\langle x, h_2^*\rangle) + \frac{\partial \phi}{\partial t_2}(\langle x, h_1^*\rangle, \langle x, h_2^*\rangle)\,(b\langle x, h_1^*\rangle + a\langle x, h_2^*\rangle),$$

and, for $y \in H$,

$$\big(D_H^2 f(x)\big)(y) = \frac{\partial^2 \phi}{\partial t_1^2}(\langle x, h_1^*\rangle, \langle x, h_2^*\rangle)\langle y, B^*h_1^*\rangle B^*h_1^* + \frac{\partial^2 \phi}{\partial t_1 \partial t_2}(\langle x, h_1^*\rangle, \langle x, h_2^*\rangle)\langle y, B^*h_2^*\rangle B^*h_1^*$$
$$+ \frac{\partial^2 \phi}{\partial t_2 \partial t_1}(\langle x, h_1^*\rangle, \langle x, h_2^*\rangle)\langle y, B^*h_1^*\rangle B^*h_2^* + \frac{\partial^2 \phi}{\partial t_2^2}(\langle x, h_1^*\rangle, \langle x, h_2^*\rangle)\langle y, B^*h_2^*\rangle B^*h_2^*.$$

Taking into account Lemma 2.2, we get

$$(2.5)\quad Lf(x) = \frac{1}{2}\sum_{i,j=1}^{2} r_{ij}\,\frac{\partial^2 \phi}{\partial t_i \partial t_j}(\langle x, h_1^*\rangle, \langle x, h_2^*\rangle) + \sum_{i,j=1}^{2} c_{ij}\ \langle x, h_j^*\rangle\ \frac{\partial \phi}{\partial t_i}(\langle x, h_1^*\rangle, \langle x, h_2^*\rangle)$$
$$= (L_2\phi)(\langle x, h_1^*\rangle, \langle x, h_2^*\rangle)).$$

Now observe that $\sigma(C) = \{a \pm ib\} \subset \mathbb{C}_-$, and the kernel of R does not contain any invariant subspace of C^*: by (2.4) this would imply degeneracy of R_∞ and Q_∞, and consequently of μ_∞. Thus we can use [9, Thm 5.1] to conclude that for any $\lambda \in \mathbb{C}_-$ there exists an eigenfunction $\phi_\lambda(t) \in L^1(\mathbb{R}^2, \nu_\infty)$ of L_2 corresponding to λ. Approximating ϕ_λ by $\phi_n \in \mathcal{C}(L_2)$, $\phi_n \to \phi_\lambda$ in $L^1(\mathbb{R}^2, \nu_\infty)$, and then applying (2.5) to $f_n(x) := \phi_n(\langle x, h_1^*\rangle, \langle x, h_2^*\rangle) \in \mathcal{C}(L)$, we obtain $f_n(x) \to \phi_\lambda(\langle x, h_1^*\rangle, \langle x, h_2^*\rangle) =: f_\lambda(x)$ in $L^1(E, \mu_\infty)$, $Lf_n(x) \to \lambda f_\lambda(x)$ in $L^1(E, \mu_\infty)$. Hence $f_\lambda(x) \in \mathcal{D}(L)$ and $Lf_\lambda(x) = \lambda f_\lambda(x)$. □

References

[1] Z. Brzeźniak, J.M.A.M. van Neerven, *Stochastic convolution in separable Banach spaces and the stochastic linear Cauchy problem*, Studia Math. **143** (2000), pp. 43–74.

[2] A. Chojnowska–Michalik, *On $L^1(H,\mu)$–properties of Ornstein-Uhlenbeck semigroups*, Stochastic partial differential equations and applications—VII, Lect. Notes Pure Appl. Math., 245, Chapman & Hall/CRC, Boca Raton, FL, 2006, pp. 77—88.

[3] A. Chojnowska–Michalik, B. Goldys, *Existence, uniqueness and invariant measures for stochastic semilinear equations on Hilbert spaces*, Probab. Theory Relat. Fields **102** (1995), pp. 331–356.

[4] A. Chojnowska–Michalik, B. Goldys, *On regularity properties of nonsymmetric Ornstein-Uhlenbeck semigroups in L^p spaces*, Stoch. Stoch. Rep. **59** (1996), pp. 183–209.

[5] G. Da Prato, J. Zabczyk, "Second order partial differential equations in Hilbert spaces", Cambridge University Press, Cambridge, 2002.

[6] G. Da Prato, J. Zabczyk, "Stochastic equations in infinite dimensions", Cambridge University Press, Cambridge, 1992.

[7] K.J. Engel, R. Nagel, "One-Parameter Semigroups for Linear Evolution Equations", Springer-Verlag, New York, 2000.

[8] B. Goldys, J.M.A.M. Neerven, *Transition semigroups of Banach space valued Ornstein–Uhlenbeck processes*, Acta Appl. Math. **76** (2003), pp. 283–330.

[9] G. Metafune, D. Pallara, E. Priola, *Spectrum of Ornstein–Uhlenbeck operators in L^p spaces with respect to invariant measures,* J. Funct. Anal. **196** (2002), pp. 40–60.

[10] J.M.A.M. van Neerven, *Second quantization and the L^p–spectrum of nonsymmetric Ornstein-Uhlenbeck operators,* Infin. Dimens. Anal. Quantum Probab. Relat. Top. **8** (2005), pp. 473–496.
[11] J.M.A.M. van Neerven, *Uniqueness of invariant measures for the stochastic Cauchy problem in Banach spaces,* Operator theory: Advances and Applications, **127** (2001), pp. 491–517.
[12] J.M.A.M. van Neerven, E. Priola, *Norm discontinuity and spectral properties of Ornstein–Uhlenbeck semigroups,* J. Evolution Equations **5** (2005), pp. 557–576.
[13] K. Yosida, "Functional Analysis", Springer-Verlag, Berlin–Göttingen–Heidelberg, 1965.

Dept of Mathematics, California Institute of Technology, MC 253-37, 1200 E.California Blvd, Pasadena, CA 91125, US.
Telephone: (626)29-89-119
E-mail: rostysla@caltech.edu